Thoughts concerning on-orbit injection of calibration electrons through thin-target elastic scattering inside the Mu2e solenoid

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Introduction

It will fall to members of the Mu2e collaboration to prove that the spectrometer's line shape and energy scale near the electron spectrum endpoint are well understood. Any claimed observation of a conversion signal is unlikely to be accepted by our colleagues without a clear demonstration of this. An ideal calibration technique would permit the collaboration to gauge the precision, absolute energy scale, and level of systematic uncertainties associated with the spectrometer during the time that the experiment is live, recording physics data for the $\mu^- \mathrm{Al} \to e^- \mathrm{Al}$ search. Such a calibration technique might

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be realizable through the installation of a suitable electron linac. The linac could be designed around a spare Project-X cryomodule or an upgraded AØ Photoinjector.[1]

How might electrons from the linac be injected on-orbit so that they follow trajectories typical of signal electrons? In Ref. [1] I discussed a downstream injection scheme which would necessitate a reconfiguration of the calorimeter. Another possibility to consider, suggested by Yuri Kolomensky, would be to inject electrons down the solenoid axis, then scatter them on-orbit via elastic collisions in a thin target.

How well might this elastic scattering approach work? Would the number of electrons per injected bunch (in order to have one electron scatter properly) be sufficiently small that the extra noise in the spectrometer would be minimal? I discuss the feasibility of this technique in this note.

Elastic scattering

The design of the Mu2e detector exploits the limited volume of phase space occupied by signal electrons of interest, namely those with momenta ~105 MeV/c that leave the stopping target roughly 90º from the detector solenoid's central axis. As a result, a calibration scheme in which we inject electrons along a magnetic field line, parallel to the solenoid axis, will necessarily require that an electron scatter through a large angle in order to find itself following a path characteristic of signal electrons.

Nonrelativistic trajectories under the influence of inverse-square central forces

To begin our discussion, consider *non-relativistic* elastic scattering of two particles that interact through a central force. This is a situation familiar from undergraduate mechanics: particle trajectories will satisfy the equation[2]

$$\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2}+\frac{1}{r}=-\frac{\mu r^2}{J^2}F(r) \qquad [1]$$

where $\mu \equiv m_1 m_2/(m_1+m_2)$ is the reduced mass, $r \equiv |\vec{x}_2-\vec{x}_1|$ is the separation between the particles, J is magnitude of the system's (conserved) angular momentum, and $F(r)$ is

[1] See George Gollin, *On the possible construction of a Mu2e calibration linac built around a spare Project-X or ILC cryomodule*, July 25, 2008, http://www.hep.uiuc.edu/home/g-gollin/mu2e/cryomodule_linac_for_mu2e_v3.pdf.

[2] See, for example, Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems, 4th Edition*, Saunders College Publishing, New York (1995), equation 8.21 on page 297. A more detailed development can be found beginning on page 217 of *Classical Mechanics and Relativity II*: http://www.hep.uiuc.edu/home/g-gollin/Physics_326_fall_2008_lecture_notes.pdf.

the (central) force acting between the particles. (The sign of F is *negative* for an attractive force, and *positive* for a repulsive force.)

When the force experienced by one particle arises from the electrostatic (or gravitational) field of a second particle, the right side of Eqn. [1] becomes a constant, yielding an inhomogeneous harmonic oscillator equation in $1/r$. Since we are working with electron-nucleus scattering, let's define the force to be $F(r) \equiv -\kappa/r^2$, where κ is positive, and assume that the total energy E is positive.

After a small amount of algebra, Eqn. [1] is found to have the solution

$$\frac{1}{r(\theta)} = \frac{\mu\kappa}{J^2} + A\cos(\theta + \varphi)$$

where the appropriate value of A can be determined, for example, using the values of the system's total energy and angular momentum. The phase φ is arbitrary, and depends on the orientation of the coordinate axes used in the problem.

One can show that the relationship between A, J, and E is

$$\frac{AJ^2}{\mu\kappa} = \sqrt{1 + \frac{2EJ^2}{\mu\kappa^2}}\,.$$

If we define

$$\alpha \equiv \frac{J^2}{\mu\kappa} \text{ and } \varepsilon \equiv \sqrt{1 + \frac{2EJ^2}{\mu\kappa^2}}$$

we can rewrite the $r(\theta)$ equation as

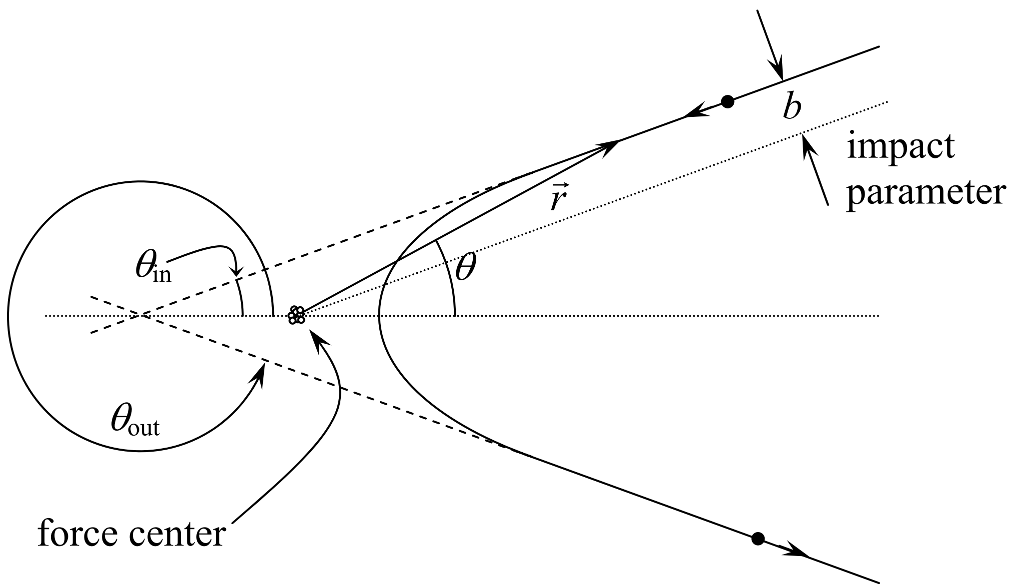
$$r(\theta) = \frac{\alpha}{1 + \varepsilon\cos(\theta)}$$

where I've picked the orientation of the coordinate axes so that $\varphi = 0$.

This is the general equation for a conic section; ε is called the *eccentricity* of the curve while α is the *semilatus rectum*. The resulting trajectories $r(\theta)$ obey this equation regardless of whether the force is attractive or repulsive.

Nonrelativistic Coulomb scattering

The following diagram illustrates the angles in the case of scattering by a repulsive force. Note that as $r \to \infty$ the angle θ approaches θ_{in}, θ_{out}. Also note the symmetry about the horizontal axis, forced by the choice $\varphi = 0$. I've indicated the particle's impact parameter b in the diagram.



The *scattering angle* θ_{scat} is not the same thing as $\theta_{\text{out}} - \theta_{\text{in}}$: a particle which is not influenced at all by the force center will have $\theta_{\text{out}} - \theta_{\text{in}} = \pi$. Rather, we need to work with the amount of change in the particle's direction of travel, which is $\theta_{scat} = \theta_{out} - \pi - \theta_{in}$. The angles θ_{in}, θ_{out} correspond to values for which the denominator in our expression for $r(\theta)$ becomes zero: $\cos\theta_{in} = \cos\theta_{out} = -1/\varepsilon$. After some algebra we find

$$\cos(\theta_{scat}) = \frac{(2Eb/\kappa)^2 - 1}{(2Eb/\kappa)^2 + 1} \qquad [2]$$

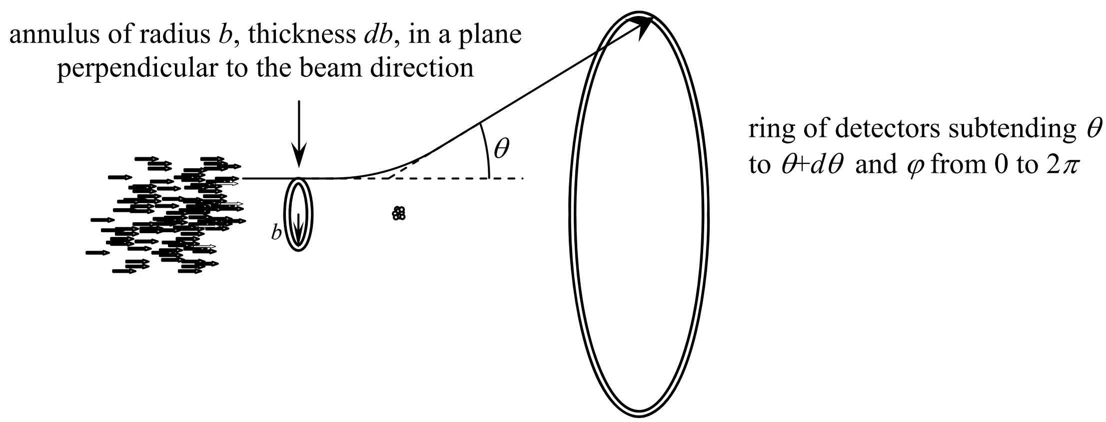
where b is the impact parameter.

We can manipulate Eqn. [2] to derive an expression for the impact parameter b as a function of scattering angle θ_{scat} :

$$b(\theta_{scat}) = \frac{\kappa}{2E}\cot\frac{\theta_{scat}}{2} \quad \text{or} \quad \theta_{scat}(b) = 2\cot^{-1}\left(\frac{2Eb}{\kappa}\right). \qquad [3]$$

Naturally, if our force didn't behave like $1/r^2$, we'd obtain a different relationship between θ_{scat} and the impact parameter *b*.

In a scattering experiment we generally do not measure the impact parameter directly. Rather, while subjecting a small target to an incident beam flux of *N* particles per unit area, we determine the relative numbers of particles that scatter at different angles θ_{scat} as shown in the following diagram.



All particles with impact parameters between *b* and *b*+*db* will scatter into the detector with scattering angles between θ and $\theta+d\theta$. Since the surface area of a ring with radius *b* and thickness *db* is $2\pi b db$ we expect that $dN = 2\pi bN db$ particles will scatter into the range θ to $\theta+d\theta$.

Differentiation of Eqns. [3] with respect to θ allows us to determine $db/d\theta$ and $dN/d\theta$. The solid angle $d\Omega$ subtended by the region between θ and $\theta+d\theta$ is $d\Omega = 2\pi \sin\theta d\theta$ so we conclude that

$$\frac{dN}{d\theta} = 2\pi N\left(\frac{\kappa}{4E}\right)^2 \frac{\sin\theta}{\sin^4(\theta/2)}$$

and

$$\frac{dN}{d\Omega} = N\left(\frac{\kappa}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}.$$

Defining the differential cross section $d\sigma/d\Omega \equiv (1/N)\,dN/d\Omega$ yields the Coulomb differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{N}\frac{dN}{d\Omega} = \left(\frac{\kappa}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}. \quad [4]$$

Note that the *total* Coulomb cross section $\sigma_{tot} = \int (d\sigma/d\Omega)\, d\Omega$ is infinite, due to the long-range nature of the Coulomb potential. (Witness the quartic divergence of the denominator at small angles.) However, thanks to the shielding of the nuclear charge by orbiting electrons, this divergence does not pose practical problems.

Geiger, Marsden, and Rutherford

Geiger, Marsden, and Rutherford observed the scattering of alpha particles emitted by radium in a thin gold foil. It is likely that the "beam" was produced by collimating 4.871 MeV alphas emitted by a sample of the naturally occurring ^{228}Ra isotope. In that the atomic number of gold is 79, we can calculate the differential cross section and impact parameter vs. angle relationships for the interactions they had observed.

Converting to S.I. units, the alpha particle energy becomes $E = 7.804\times10^{-13}$ J. The proportionality constant κ for the force acting on the alpha is $Q_{gold}Q_\alpha / 4\pi\varepsilon_0$ so that $\kappa = 3.64\times10^{-26}$ and $\kappa/E = 4.67\times10^{-14}$. With this, we can recast Eqns. [3] as

$$b(\theta_{\alpha-Au}) = 2.33\times10^{-14} \cot\frac{\theta_{\alpha-Au}}{2} \quad \text{or} \quad \theta_{\alpha-Au}(b) = 2\cot^{-1}\left(4.29\times10^{13} b\right).$$

$\theta_{\alpha-Au} = 90°$ scattering results from an impact parameter $b = 2.33\times10^{-14}\,\text{m}$, roughly three times larger than the $\sim 7\times10^{-15}\,\text{m}$ nuclear radius of ^{197}Au.[3]

Also in S.I. units, the differential cross section for alpha-gold scattering (expressed in units of m^2 since solid angle $d\Omega$ is dimensionless) is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\alpha-Au} = \frac{1.36\times10^{-28}}{\sin^4(\theta/2)}.$$

A common practical unit used in discussing relativistic scattering is the *barn* (as in "as big as a barn"[4]), with 1 barn = 10^{-28} m^2.

Relativistic modifications

The description of the scattering process changes when one or both of the colliding particles is moving at relativistic speed. Even so, relativistic dynamics preserves two of the central features of non-relativistic mechanics: the connection between impulse and force, namely $d\vec{p} = \vec{F}dt$, and the work-energy theorem, $dE = \vec{F}\cdot d\vec{x}$. (Naturally, $d\vec{p}$ is the

[3] See, for example, Landolt-Börnstein, *Nuclear Charge Radii,* Volume 20 of *Group I Elementary Particles, Nuclei and Atoms*, pp 1-4, Springer Berlin Heidelberg (2004): http://www.springerlink.com/content/tvu6741611565jj6/fulltext.pdf

[4] See "Hitting the broad side of a (classified) barn," *Symmetry Magazine*, February, 2006, for the origin of this term: http://www.symmetrymagazine.org/cms/?pid=1000258.

change in a particle's three-momentum, while dE is the change in its total energy.) However, the relativistic expression for vector momentum is $\vec{p} = \gamma m\vec{v}$ rather than $\vec{p} = m\vec{v}$. As a result, small changes in speed effect dramatic changes in γ for a fast particle so that force is no longer proportional to acceleration. In addition, a force applied perpendicular to a particle's velocity drives changes in $\gamma m\vec{v}$ rather than $m\vec{v}$ so the transverse deflection that results is smaller by a factor of γ than in the non-relativistic case.[5]

Combining all the relativistic effects reveals this connection between applied force and change in three-vector velocity:

$$\vec{a} = \begin{bmatrix} a_{\parallel} \\ \vec{a}_{\perp} \end{bmatrix} \equiv \begin{bmatrix} dv_{\parallel}/dt \\ d\vec{v}_{\perp}/dt \end{bmatrix} = \frac{1}{\gamma m}\begin{bmatrix} 1/\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F_{\parallel} \\ \vec{F}_{\perp} \end{bmatrix}.$$

Naturally, the subscripts $\parallel$ and $\perp$ refer to components of the three-vectors parallel and perpendicular to the particle's velocity.

The different force-acceleration relationship modifies the scattering process so that it no longer exhibits the form found by Rutherford. In addition, the quantum mechanics of scattering (in which we must average over incoming electron polarizations and sum over outgoing electron polarizations) plays a role. The correct relativistic expression for an electron scattering off a heavy point target is the Mott scattering cross section:

$$\left.\frac{d\sigma}{d\Omega}\right|_{Mott} = \left(\frac{\kappa}{2p^2c^2\sin^2(\theta/2)}\right)^2\left[m^2c^4 + p^2c^2\cos^2(\theta/2)\right].$$

In this expression $p \equiv |\vec{p}| = |\gamma m\vec{v}|$. Note that the Mott scattering formula ignores the effects of nuclear recoil.

In the nonrelativistic limit where $p^2c^2 \ll m^2c^4$ we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{Mott} \to \left(\frac{\kappa mc^2}{2m^2v^2c^2\sin^2(\theta/2)}\right)^2 = \left(\frac{\kappa}{4E}\right)^2\frac{1}{\sin^4(\theta/2)},$$

which is just the nonrelativistic scattering cross section described previously.

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[5] Again, see appropriate sections of *Thornton and Marion*, or my development beginning on page 395 of *Classical Mechanics and Relativity II*: http://www.hep.uiuc.edu/home/g-gollin/Physics_326_fall_2008_lecture_notes.pdf.

Hyper relativistic Coulomb scattering

Calibration electrons that would be of interest to Mu2e will be traveling close to the speed of light. In this case the Mott cross section simplifies: $E^2 = p^2c^2 + m^2c^4 \approx p^2c^2$ so that

$$\left.\frac{d\sigma}{d\Omega}\right|_{Mott} \approx \left(\frac{\kappa\cos(\theta/2)}{2E\sin^2(\theta/2)}\right)^2$$

as long as $\cos(\theta/2)$ is not so small as to make the approximation $m^2c^4 + p^2c^2\cos^2(\theta/2) \approx p^2c^2\cos^2(\theta/2)$ invalid, as would happen for $\theta \approx 180°$.

We can reverse-engineer the expression for the cross section to determine the relationship between scattering angle and impact parameter. Since the scattering angle decreases monotonically with impact parameter, we can write

$$\pi\left[b(\theta_{\min})\right]^2 = \int_{\theta=\theta_{\min}}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\frac{d\sigma}{d\Omega}d\Omega = \int_{\theta=\theta_{\min}}^{\theta=\pi} 2\pi\sin\theta\frac{d\sigma}{d\Omega}d\theta\,.$$

The integral, using the approximation for the Mott scattering cross section, can be evaluated without difficulty to yield

$$b(\theta) = \frac{\kappa}{E}\sqrt{\frac{1}{\sin^2(\theta/2)} + 2\ln(\sin(\theta/2)) - 1}\,. \qquad [5]$$

In the case $\theta = 90°$ the square root evaluates to 0.554 so that $b(90°) = 0.554\,\kappa/E$.

It is amusing to see how similar the Mott cross section calculation of impact parameter for a 90° scatter is to the result for nonrelativistic scattering at the same angle, as represented by the first of Eqns.[3]: $b(\theta_{scat}) = \frac{\kappa}{2E}\cot\frac{\theta_{scat}}{2} \Rightarrow b(90°) = 0.5\,\kappa/E$. Bear in mind that the energy E in the nonrelativistic expression represents only the particle's kinetic energy, not its total energy $\gamma mc^2 \approx mc^2 + mv^2/2$.

Nuclear effects

The Mott scattering formula describes electron scattering from a heavy point charge, ignoring the recoil of the nucleus as well as any softening of the collision due to the non-pointlike charge distribution of the nucleus.

Bjorken and Drell discuss the recoil correction in *Relativistic Quantum Mechanics*[6], showing that the correction to the scattering cross section can be written this way:

$$\frac{d\sigma}{d\Omega}=\frac{E'/E}{1+(2E/M)\sin^2(\theta/2)}\times\left.\frac{d\sigma}{d\Omega}\right|_{Mott}.$$

In this expression E and E' are the initial and final electron energies while M is the mass of the target nucleus.

For gold (atomic weight 197) the nuclear mass is M = 184.79 GeV/c^2. As a result, the $2E/M$ term is responsible for a correction that is typically 0.1% at most. The E'/E numerator is nearly unity: a gold nucleus recoiling with momentum 105 MeV/c carries off only 30 keV so that the energy lost by a 105 MeV electron is ~0.03% of its original energy. As a result, it is safe to ignore recoil effects.

The finite size of the nucleus also bears consideration. The radius of a gold nucleus is about 7 fermis; if the electron's impact parameter is appreciably smaller than this some modification of the scattering cross section will be necessary. I had showed that the impact parameter for a 90° scatter is $b(90°)=0.554\,\kappa/E$ where $\kappa=Q_{gold}Q_{beam}/4\pi\varepsilon_0\approx1.82\times10^{-26}$ for an electron beam and $E = 1.68\times10^{-11}$ J for an electron energy of 105 MeV. The (point source) Mott scattering calculation yields $\kappa/E\approx1.08\times10^{-15}$ so that $b(90°)\approx0.6\times10^{-15}\,\text{m},$ much smaller than the nuclear radius. For an angle of 60° the required impact parameter is still small, approximately 1.3 fermis.

This would suggest that a gold nucleus does not have the single-scattering power to effect such a large change in direction of a 105 MeV electron. Eqn. [5] reveals that a scattering angle of 15.5° corresponds to an impact parameter of 7.1 fermis. Significantly larger angles, corresponding to smaller impact parameters, require the electron to pass through the nucleus. Clearly, the cross section to scatter through an angle at least as large as 60° will be considerably smaller than $\pi\left[b(60°)\right]^2\approx5.9\times10^{-30}\,\text{m}^2=.059$ barns.

Radiative processes

I haven't included an estimate of energy loss due to radiation by the electron during the scattering process.

In my experience, electrodynamic processes such as ILC beamstrahlung are described surprisingly well with classical, Jackson-style estimates for the radiation. It is quite likely that deflecting a 105 MeV/c electron through 60 degrees in the hundred fermi path length

[6] James D. Bjorken and Sidney D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, page 115 (1964).

through which most of the scattering impulse is applied will tend to inflict significant radiative energy loss on the incident electron.

Since the Mott cross section is calculated for a collision in which no radiation is emitted, it is likely that the large-angle scattering-with-radiation cross section will be larger than the Mott cross section. If this is the case it will introduce uncertainty in the actual energy of any large-angle scatters that are observed in the spectrometer.

Thin target scattering

Let's assume that a .001 radiation length gold foil target is used to scatter 105 MeV electrons, and that $\theta \geq 60°$ scattering cross section is $\sigma_{\geq 60°} = 5 \times 10^{-30}\,\mathrm{m}^2$. What is the scattering rate per electron incident on the target?

The radiation length of gold is $X_0 = 6.46\ \mathrm{g \cdot cm^{-2}} = 64.6\ \mathrm{kg \cdot m^{-2}}$ and its density is $\rho = 19.32\ \mathrm{g \cdot cm^{-3}} = 19.32\ \times 10^3\ \mathrm{kg \cdot m^{-3}}$.[7] As a result, a .001 radiation length foil will have thickness $t = 10^{-3} X_0 / \rho = 3.34\ \mu\mathrm{m}$. The number of gold nuclei (each of which presents a scattering cross section of .059 barns) in a one square meter sheet of gold that is 3.34 µm thick is $N = \rho V N_A \times 1000 / A_{Au} = 19.32 \times 10^3 \times 3.34 \times 10^{-6} \times 6.022 \times 10^{26} / 196.97$ $= 1.97 \times 10^{23}$. (N_A is Avogadro's Number.) With a cross section $\sigma_{\geq 60°} = 5 \times 10^{-30}\,\mathrm{m}^2$, the probability that a single electron will scatter while passing through the gold foil is $P_{\geq 60°} = 1.97 \times 10^{23} \times 5 \times 10^{-30} = 0.99 \times 10^{-6} \approx 10^{-6}$.

Calibration running

The most useful mode for accumulating calibration data will be to inject electrons periodically during actual data taking, with the period adjusted so that only a small fraction of muon beam bursts include a calibration shot from the linac.

One candidate linac that might become available is the AØ photoinjector, which could be upgraded to provide electrons of sufficiently high energy. Currently AØ can run at 10 Hz; if this rep rate were to be used for Mu2e, an intensity of 10^5 delivered electrons per linac shot would provide about one $\geq 60°$ calibration electron per second. (The reduction in $\geq 60°$ scattering rate due to the finite nuclear size would necessitate an even higher electron flux.)

This seems like an uncomfortable large rate to me. It is a concern that the high instantaneous intensity from the linac might alter the detector environment sufficiently to complicate the use of calibration electrons in determining the spectrometer line shape.

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[7] "Review of Particle Properties," Particle Data Group, *Physics Letters B*, 667, p. 110 (September 18, 2008)

Conclusion

Elastic scattering of 105 MeV electrons from a .001 X_0 gold target inside the Mu2e solenoid is unlikely to provide a sufficiently large number of calibration electrons to the collaboration to resolve the complex problems of determining line shape during actual physics running. The small scattering cross section would require more than 10 TeV of energy, carried by a sub-nanosecond bunch of well over 10^5 electrons, to be injected into the detector solenoid at 10 Hz. This is likely to complicate the comparison of the calibration environment with the standard data taking environment at Mu2e.